Application of Variational Iteration Method for *n*th-Order Integro-Differential Equations

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In this paper, the variational iteration method is proposed to solve Fredholm's *n*th-order integro-differential equations. The initial approximation is selected wisely which satisfies the initial conditions. The results reveal that this method is very effective and convenient in comparison with other methods.

Key words: Variational Iteration Method; Homotopy Perturbation Method; Differential Transformation Method; Integro-Differential Equation; Volterra's Integro-Differential Equation.

1. Introduction

The variational iteration method [1, 2], which is a modified general Lagrange multiplier method, has been shown to solve effectively, easily, and accurately a large class of nonlinear problems with approximations which converges (locally) to accurate solutions (if certain Lipschitz-continuity conditions are met). It was successfully applied to autonomous ordinary differential equations and nonlinear partial differential equations with variable coefficients [3], to Schrödinger-KdV, generalized KdV and shallow water equations [4], to Burgers' and coupled Burgers' equations [5], to the linear Helmholtz partial differential equation [6], and recently to nonlinear fractional differential equations with Caputo differential derivative [7], and other fields [8 – 10]. Also, the variational iteration method is applied to fourth-order Volterra's integro-differential equations [11] and J. H. He used it for solving some integro-differential equations [12] by choosing the initial approximate solution in the form of a exact solution with unknown constants. On the other hand, Golbabai and Javidi solved the *n*th-order integro-differential equations [13] by transforming to a system of ordinary differential equations and using the homotopy method.

The purpose of this paper is to extend the analysis of the variational iteration method for solving the general *n*th-order integro-differential equations as follows:

$$y^{(n)}(x)+f(x)y(x)+\int_a^b w(x,t)y^{(m)}(t)\mathrm{d}t=g(x), \quad a<x<b, \tag{1}$$

with initial conditions

$$\begin{aligned} &y(a)=\alpha_0,\\ &y'(a)=\alpha_1,\ y''(a)=\alpha_2,\ \ldots,\ y^{(n-1)}(a)=\alpha_{n-1}, \end{aligned} \tag{2}$$

where α_i, $i=0,1,\ldots,n-1$, are real constants, m and n are integers with $m<n$. In (1) the functions f, g and w are given, and the solution y should be determined. We assume that (1) has a unique solution. In addition, we compare results with other methods in Section 3. It is shown that this method is very simple and effective. Finally one conclusion is stated in Section 4.

2. Variational Iteration Method

To illustrate the basic concept of the variational iteration method, we consider the following general nonlinear system:

$$L[y(x)]+N[y(x)]=\psi(x),$$

where L is a linear operator, N is a nonlinear operator and $\psi(x)$ is a given continuous function. According to the variational iteration method [8, 14, 15], we can construct a correction functional in the form

$$y_{k+1}(x)=y_k(x)+\int_0^x\lambda(s)[Ly_k(s)+N\tilde{y}_k(s)-\psi(s)]\mathrm{d}s,$$

where $y_0(x)$ is an initial approximation with possible unknowns, λ is a Lagrange multiplier which can be identified optimally via variational theory, the subscript k denotes the kth approximation, and $\tilde{y}_k$ is considered as a restricted variation [8], i. e. $\delta\tilde{y}_k = 0$. It is shown this method is very effective and easy for a linear problem, its exact solution can be obtained by only one iteration, because λ can be exactly identified. It should be specially pointed out that the variational iteration method is a powerful method for engineering applications [16–22].

For solving (1) by the variational iteration method, for simplicity, we consider all terms as restricted variations except $y^{(n)}(x)$. According to the variational iteration method, we derive a correction functional as follows:

$$y_{k+1}(x) = y_k(x) + \int_0^x \lambda(s)\Big[y_k^{(n)}(s) + f(s)\tilde{y}_k(s) + \int_a^b w(s,t)\tilde{y}_k^{(m)}(t)\mathrm{d}t - g(s)\Big]\mathrm{d}s$$

and the stationary condition of the above correction functional can be expressed as

$$\begin{aligned}
&\lambda^{(n)}(s) = 0,\\
&1 + (-1)^{n-1}\lambda^{(n-1)}(s)|_{s=x} = 0,\\
&\lambda^{(i)}(s)|_{s=x} = 0,\\
&\lambda(s)|_{s=x} = 0,\\
&\text{with } i = 1,2,\dots,n-2.
\end{aligned}$$

The Lagrange multiplier, therefore, can be identified as

$$\lambda(s) = \frac{(-1)^n}{(n-1)!}(s-x)^{n-1}$$

and as a result, we obtain the following iteration formula:

$$y_{k+1}(x) = y_k(x) + \int_0^x \frac{(-1)^n}{(n-1)!}(s-x)^{n-1}\Big[y_k^{(n)}(s) + f(s)y_k(s) + \int_a^b w(s,t)y_k^{(m)}(t)\mathrm{d}t - g(s)\Big]\mathrm{d}s. \tag{3}$$

3. Applications

In this section, we present some examples to show efficiency and high accuracy of the variational iteration method for solving (1).

Example 3.1. Let us first consider the integro-differential equation

$$y'(x) = 1 - \frac{1}{3}x + \int_0^1 xty(t)\mathrm{d}t, \quad y(0) = 0,$$

with the exact solution

$$y(x) = x.$$

According to (3) we have the following iteration formulae:

$$y_{k+1}(x) = y_k(x) - \int_0^x \Big[y_k'(s) - 1 + \frac{1}{3}s - \int_0^1 sty_k(t)\mathrm{d}t\Big]\mathrm{d}s.$$

Now, we choose the initial approximation $y_0 = 0$ which satisfies the initial condition. Then we obtain

$$\begin{aligned}
y_1 &= x - \frac{1}{6}x^2,\\
y_2 &= x - \frac{1}{48}x^2,\\
y_3 &= x - \frac{1}{384}x^2,\\
y_4 &= x - \frac{1}{3072}x^2,\\
y_5 &= x - \frac{1}{24576}x^2,\\
y_{10} &= x - \frac{1}{805306368}x^2,\\
y_{16} &= x - \frac{1}{211106232532992}x^2.
\end{aligned}$$

It is obvious that the iterations converge to the exact solution and the results are exactly the same that were obtained with the homotopy perturbation method [13]. We can see it does not need to transform into the system of ordinary differential equations and also, it is applied very convenient.

Example 3.2. Consider the problem with $n = 2$ and $m = 1$ as follows:

$$y''(x) = \mathrm{e}^x - x + \int_0^1 xty(t)\mathrm{d}t, \quad y(0) = 1,\ y'(0) = 1,$$

with the exact solution

$$y(x) = \mathrm{e}^x.$$

According to (3) we have the following iteration formulation:

$$y_{k+1}(x) = y_k(x) + \int_0^x (s-x)\Big[y_k''(s) - \mathrm{e}^s + s - \int_0^1 sty_k(t)\mathrm{d}t\Big]\mathrm{d}s$$

with the initial approximation $y_0 = x+1$, which satisfies the initial conditions. Then we will have the below approximations:

$$y_1(x) = e^x - \frac{1}{36}x^3,$$
$$y_2(x) = e^x - \frac{1}{1080}x^3,$$
$$y_3(x) = e^x - \frac{1}{32400}x^3,$$
$$y_4(x) = e^x - \frac{1}{972000}x^3,$$
$$y_5(x) = e^x - \frac{1}{29160000}x^3,$$
$$y_{10}(x) = e^x - \frac{1}{708588000000000}x^3.$$

It is obvious that the iterations converge to the exact solution and we can see that the above results are better than the results obtained from homotopy method [13]. As example, the comparisons of the two methods for some iterations:

$$y^5_{\text{hom}}(x) = \sum_{i=1}^{5} v_i = e^x - \frac{1}{1080}x^3,$$
$$y^8_{\text{hom}}(x) = \sum_{i=1}^{8} v_i = e^x - \frac{1}{32400}x^3,$$
$$y^{10}_{\text{hom}}(x) = \sum_{i=1}^{10} v_i = e^x - \frac{1}{972000}x^3.$$

Example 3.3. Consider the third-order integro-differential equation

$$y'''(x) = \sin(x) - x + \int_0^{\frac{\pi}{2}} xty'(t)\mathrm{d}t,$$
$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1,$$

with the exact solution

$$y(x) = \cos(x).$$

Using the variational iteration method (3) with the initial approximation $y_0(x) = -\frac{1}{2}x^2 + 1$ which satisfies initial conditions, gives

$$y_{k+1}(x) = y_k(x) - \int_0^x \frac{1}{2}(s-x)^2 \Big[y_k'''(s) - \sin(s) + s - \int_0^{\frac{\pi}{2}} sty_k'(t)\mathrm{d}t \Big]\mathrm{d}s.$$

Table 1. Error of numerical results for Example 3.3.

x	HPM, $N=5$	VIM, $N=5$	HPM, $N=8$	VIM, $N=8$
0.2	7.4074e-6	2.0095e-7	2.4691e-8	6.5092e-9
0.4	5.9259e-5	3.2152e-6	1.9753e-7	1.0414e-7
0.6	2.0000e-4	1.6277e-5	6.6667e-7	5.2725e-7

Then, we have the following primary approximations:

$$y_1(x) = \cos(x) + \frac{1}{576}\pi^3 x^4 - \frac{1}{24}x^4,$$
$$y_2(x) = \cos(x) + \frac{1}{23040}x^4\pi^5 - \frac{1}{552960}x^4\pi^8,$$
$$y_3(x) = \cos x + \frac{1}{530841600}\pi^{13}x^4 - \frac{1}{22118400}\pi^{10}x^4,$$
$$y_4(x) = \cos(x) + \frac{1}{21233664000}x^4\pi^{15} - \frac{1}{509607936000}x^4\pi^{18},$$
$$y_5(x) = \cos(x) + \frac{1}{489223618560000}x^4\pi^{23} - \frac{1}{20384317440000}x^4\pi^{20},$$
$$y_6(x) = \cos(x) - 0.0000400364400x^4,$$
$$y_7(x) = \cos(x) + 0.00001276243625x^4,$$
$$y_8(x) = \cos(x) - 0.00000406828825x^4.$$

In order to show the efficiency and high accuracy of the presented method we report the absolute error which is defined by

$$Ey_N(x) = |y_{\text{exact}}(x) - y_{N\text{vim}}(x)|.$$

In Table 1, we listed the results obtained by variational iteration method (VIM) compared with those given by the homotopy perturbation method (HPM). As we see from Table 1, it is clear that the results obtained by the presented method are very superior to that obtained by HPM. Also, the perform of VIM method is very simple.

Example 3.4. Consider the integro-differential equation

$$y'(x) = xe^x + e^x - x + \int_0^1 xy(t)\mathrm{d}t, \quad y(0) = 0,$$

with the exact solution

$$y(x) = xe^x.$$

Using the variational iteration method (3) with the initial approximation $y_0(x) = x$ which satisfies the initial conditions, gives

$$y_{k+1}(x) = y_k(x) - \int_0^x \Big[y_k'(s) - s\mathrm{e}^s - \mathrm{e}^s + s - \int_0^1 s y_k(t)\mathrm{d}t\Big]\mathrm{d}s.$$

Now, we can obtain the following approximations:

$$y_1(x) = x\mathrm{e}^x - \frac{1}{4}x^2,$$
$$y_2(x) = x\mathrm{e}^x - \frac{1}{24}x^2,$$
$$y_3(x) = x\mathrm{e}^x - \frac{1}{144}x^2,$$
$$y_4(x) = x\mathrm{e}^x - \frac{1}{864}x^2,$$
$$y_5(x) = x\mathrm{e}^x - \frac{1}{5184}x^2,$$
$$y_6(x) = x\mathrm{e}^x - \frac{1}{31104}x^2,$$
$$y_7(x) = x\mathrm{e}^x - \frac{1}{186624}x^2,$$
$$y_8(x) = x\mathrm{e}^x - \frac{1}{1119744}x^2,$$
$$y_9(x) = x\mathrm{e}^x - \frac{1}{6718464}x^2,$$
$$y_{10}(x) = x\mathrm{e}^x - \frac{1}{40310784}x^2.$$

The absolute error of the results is given in Table 2. We also compared our results with the results obtained by the differential transformation method (DTM) in [23]. It is clear that the results obtained by the presented method are very superior to that obtained by DTM. In addition, the performance of the VIM method is very simple.

Table 2. Error of numerical results for Example 3.4.

x	Differential Transformation Method (DTM) $N = 10$	VIM $N = 10$
0.0	0.00000000e+00	0.00000000e+00
0.1	1.00118319e-02	2.48072575e-10
0.2	2.78651355e-02	9.92290301e-10
0.3	5.08730892e-02	2.23265317e-09
0.4	7.55356316e-02	3.96916120e-09
0.5	9.71888593e-02	6.20181438e-09
0.6	1.09551714e-01	8.93061271e-09
0.7	1.04133232e-01	1.21555561e-08
0.8	6.94512700e-02	1.58766448e-08
0.9	1.00034260e-02	2.00938786e-08
1.0	1.55147712e-01	2.48072575e-08

Example 3.5. Consider the second-order integro-differential equation

$$y''(x) + xy'(x) - xy(x) = \mathrm{e}^x - 2\sin(x) + \int_{-1}^1 \sin(x)\mathrm{e}^{-t}y(t)\mathrm{d}t,$$
$$y(0) = 1, \quad y'(0) = 1,$$

with the exact solution

$$y(x) = \mathrm{e}^x.$$

Using the variational iteration method (3) with the initial approximation $y_0(x) = x + 1$ which satisfies the initial conditions, gives

$$y_{k+1}(x) = y_k(x) + \int_0^x (s-x)\Big[y_k''(s) + sy_k'(s) - sy_k(s) - \mathrm{e}^s + 2\sin(s) - \int_{-1}^1 \sin(s)\mathrm{e}^{-t}y_k(t)\mathrm{d}t\Big]\mathrm{d}s.$$

Then, we have the following primary approximations:

$$y_1(x) = -2x + x\mathrm{e}^1 - 3x\mathrm{e}^{-1} + 2\sin(x) + \mathrm{e}^x + 1/12x^4 - \mathrm{e}^1\sin(x) + 3\mathrm{e}^{-1}\sin(x),$$

$$\begin{aligned}
y_2(x) = {} & 4 - 1/4x\mathrm{e}^1 - x\mathrm{e}^{-1}\cos(1) - x\mathrm{e}^{-1}\sin(1) \\
& -3/2x\mathrm{e}^{-2}\cos(1) - 3/2x\mathrm{e}^{-2}\sin(1) - x\mathrm{e}^1\sin(1) \\
& +1/2x\mathrm{e}^2\sin(1) - 1/2x\mathrm{e}^2\cos(1) + 6x\mathrm{e}^{-2} - x\sin(1) \\
& -2\mathrm{e}^1 + 6\mathrm{e}^{-1} + 2x\cos(1) - 2\sin(x)x - 6\mathrm{e}^{-1}\cos(x) \\
& +2\cos(x)x + \sin(1)\sin(x) + 2\mathrm{e}^1\cos(x) - 6\mathrm{e}^{-2}\sin(x) \\
& -2\cos(1)\sin(x) + 19/12x\mathrm{e}^{-1} - 1/6x^4 - 4\cos(x) \\
& -2\sin(x) + \mathrm{e}^x + 5/4\mathrm{e}^1\sin(x) - 55/12\mathrm{e}^{-1}\sin(x) \\
& +1/12x^4\mathrm{e}^1 + \cos(1)\mathrm{e}^{-1}\sin(x) + \sin(1)\mathrm{e}^{-1}\sin(x) \\
& +\sin(1)\mathrm{e}^1\sin(x) + 1/2\mathrm{e}^2\cos(1)\sin(x) \\
& +3/2\mathrm{e}^{-2}\cos(1)\sin(x) + 3/2\mathrm{e}^{-2}\sin(1)\sin(x) \\
& -\cos(1)\mathrm{e}^1\sin(x) - 1/2\mathrm{e}^2\sin(1)\sin(x) - 3\mathrm{e}^{-1}\sin(x)x \\
& +1/3x^3 - 1/90x^6 + 1/504x^7 - 1/4\mathrm{e}^{-1}x^4 \\
& +1/2\mathrm{e}^{-1}x^3 - 1/6\mathrm{e}^1x^3 + x\mathrm{e}^1\cos(1) + \mathrm{e}^1\sin(x)x \\
& -x\cos(x)\mathrm{e}^1 + 3x\cos(x)\mathrm{e}^{-1}.
\end{aligned}$$

This example has been solved by the homotopy perturbation method in [13]. In order to show the efficiency and high accuracy of the presented method, in Figure 1, we plotted the error functions, i. e.

$$E_N^{\mathrm{vim}}(x) = y_{\mathrm{exact}} - y_N^{\mathrm{vim}}(x),$$

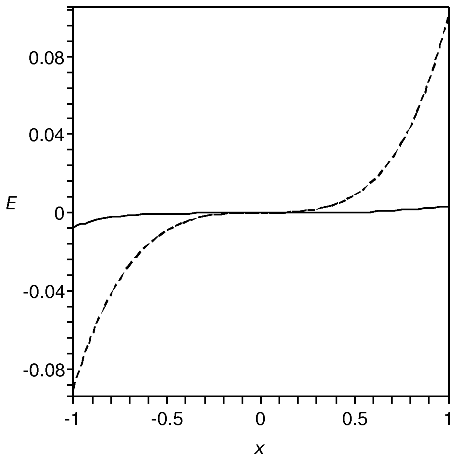

Fig. 1. The error function with $N = 3$ in interval $[-1, 1]$, solid line: variational iteration method; dashed line: homotopy perturbation method.

$$E_N^{\text{hpm}}(x) = y_{\text{exact}} - \sum_{k=1}^{N} y_k^{\text{hpm}}(x),$$

where N denotes the numbers of iterations.

As we see in Figure 1, the error of this method with the same iterations is less than that with the homotopy perturbation method.

Example 3.6. Consider the second-order integro-differential equation

$$y''(x) + xy(x) = -(1+x)\cos(x) - \frac{1}{2}(\mathrm{e}^x(\cos(x) + \sin(x)) - 1)x^2 + \int_0^x x^2 \mathrm{e}^t y(t)\mathrm{d}t, \;\; y(0) = 1, y'(0) = 0,$$

with the exact solution

$$y(x) = \cos(x).$$

According to (3) we have the following iteration formulation:

$$y_{k+1}(x) = y_k(x) + \int_0^x (s-x)\Big[y_k''(s) + sy_k(s) + (1+s)\cos(s) + \frac{1}{2}(\mathrm{e}^s(\cos(s) + \sin(s)) - 1)s^2 - \int_0^s s^2 \mathrm{e}^t y_k(t)\mathrm{d}t\Big]\mathrm{d}s,$$

with the initial approximation $y_0 = 1$, which satisfies the initial conditions, we will have the below approximations:

$$\begin{aligned} y_1(x) = & -27/4 - 3/2x - \mathrm{e}^x\cos(x)x \\ & -1/4\,\mathrm{e}^x\sin(x)x^2 + 1/4\,\mathrm{e}^x\cos(x)x^2 - 1/6x^3 \\ & -1/24x^4 + 3/4\,\mathrm{e}^x\cos(x) + 3/4\,\mathrm{e}^x\sin(x) + \cos(x)x \\ & + \mathrm{e}^x x^2 + 6\,\mathrm{e}^x - 2\sin(x) + \cos(x) - 4\,\mathrm{e}^x x, \end{aligned}$$

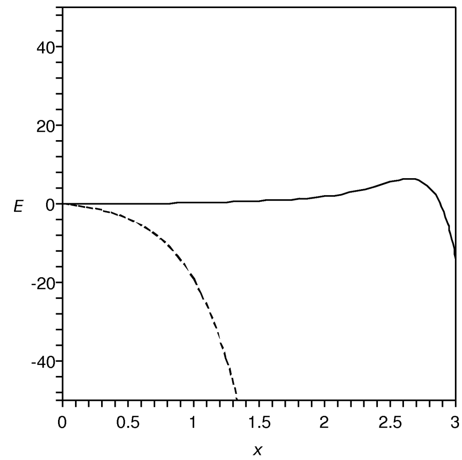

Fig. 2. The error function with only $N = 2$ in interval $[0, 3]$, solid line: variational iteration method; dashed line: homotopy perturbation method.

$$\begin{aligned} y_2(x) = & \, 57967/5000x - 1/4\,\mathrm{e}^x\cos(x)x \\ & +7/4\,\mathrm{e}^x\sin(x)x^2 + 9/4\,\mathrm{e}^x\cos(x)x^2 \\ & +1657/2500\,\mathrm{e}^{2x}\sin(x)x^2 - 2284/3125\,\mathrm{e}^{2x}\sin(x)x \\ & -29/4\,\mathrm{e}^x\sin(x)x + 13/500\,\mathrm{e}^{2x}\cos(x)x^4 \\ & -3/8\,\mathrm{e}^x\cos(x)x^3 + 1/8\,\mathrm{e}^x\sin(x)x^3 + 141/32\,\mathrm{e}^{2x} \\ & +9/8x^3 + 589/6000x^4 - 21/4\,\mathrm{e}^x\cos(x) \\ & +21/4\mathrm{e}^x\sin(x) + 2\cos(x)x - 245/4\,\mathrm{e}^x x^2 - 291/2\,\mathrm{e}^x \\ & -4\sin(x) - 9\cos(x) + 134\,\mathrm{e}^x x + 1/180x^6 + 1/1008x^7 \\ & -21/250\,\mathrm{e}^{2x}\cos(x)x^3 - 301/2500\,\mathrm{e}^{2x}\cos(x)x^2 \\ & +1812/3125\,\mathrm{e}^{2x}\cos(x)x + 9/500\,\mathrm{e}^{2x}\sin(x)x^4 \\ & -53/250\,\mathrm{e}^{2x}\sin(x)x^3 + \cos(x)x^2 + 35/2\,\mathrm{e}^x x^3 \\ & -15/4\mathrm{e}^x x^4 - 16101/31250\,\mathrm{e}^{2x}\cos(x) - 1/24\,\mathrm{e}^x x^6 \\ & -6\sin(x)x + 1/2\,\mathrm{e}^x x^5 + 7257/31250\,\mathrm{e}^{2x}\sin(x) \\ & +78429491/500000 + 1/8\,\mathrm{e}^{2x}x^4 - 9/8\mathrm{e}^{2x}x^3 \\ & +65/16\,\mathrm{e}^{2x}x^2 - 103/16\,\mathrm{e}^{2x}x. \end{aligned}$$

As the same of before example, the errors of VIM and HPM are showed in Figure 2 for the case of $N = 2$.

4. Conclusions

In this paper, we have studied the *n*th-order integro-differential equations with the variational iteration method. The initial approximation was selected wisely not in form of the exact solution with unknown constants. The results showed that the variational iteration method is remarkably effective and it is very easy. In addition, it has more accuracy than the homotopy method and the differential transformation method.

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